~~VE~~ $2xy'' + 5y' + xy = 0$ USING A "MACLAURIN" SERIES

↳ ABOUT $x=0$

(IS A SINGULAR POINT OF DE)

$$y'' + \frac{5}{2x}y' + \frac{1}{2}y = 0$$

$p(x) = \frac{5}{2x}$ $\quad xp(x) = \frac{5}{2}$

$q(x) = \frac{1}{2}$ $\quad x^2q(x) = \frac{1}{2}x^2$

BOTH ANALYTIC @ $x=0$ → $x=0$ IS A REGULAR SINGULAR POINT

LET $y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' + 5y' + xy = 2x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$+ 5\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$+ x\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$2r(r-1)+5r=0 \quad \text{or} \quad a_0=0$$

$$2r^2-2r+5r=0$$

$$2r^2+3r=0$$

$$r(2r+3)=0$$

$$r=0,-\tfrac{3}{2}$$

$(r=0)$

IF $a_0=0$ AND $a_1=1$

$$(2(1)0+5(1))1=0 \quad \times$$

NO SUCH CASE

$$a_n=-\frac{1}{n(2n+3)}a_{n-2}$$

FOR $n\geq 2$

LET $a_0=1, a_1=0 \rightarrow a_3=a_5=a_7=\cdots=0$

$n=2: a_2=-\dfrac{1}{2\cdot 7}a_0=-\dfrac{1}{2\cdot 7}$ $\quad n=1$

$n=4: a_4=-\dfrac{1}{4\cdot 11}a_2=\dfrac{1}{(2\cdot 4)(7\cdot 11)}$ $\quad n=2$

$n=6: a_6=-\dfrac{1}{6\cdot 15}a_4=-\dfrac{1}{(2\cdot 4\cdot 6)(7\cdot 11\cdot 15)}$ $\quad n=3$

$$a_{2n}=\frac{(-1)^n}{(2\cdot 4\cdot 6\cdots 2n)(7\cdot 11\cdot 15\cdots(4n+3))}=\frac{(-1)^n}{2^n n!\,(7\cdot 11\cdot 15\cdots(4n+3))}$$

$$y_1=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n!\,(7\cdot 11\cdot 15\cdots(4n+3))}x^{2n}$$

$$= -\frac{3}{2} : \quad a_n = -\frac{1}{(n-\frac{3}{2})(2n+\cancel{2(\frac{-3}{2})+3})} a_{n-2} \quad \text{FOR } n \geq 2$$

$$= -\frac{1}{(n-\frac{3}{2})\, 2n} a_{n-2}$$

$$= -\frac{1}{n(2n-3)} a_{n-2}$$

LET $a_0 = 1, \ a_1 = 0$

$\hookrightarrow a_3 = a_5 = a_7 = \cdots = 0$

$$=2: a_2 = -\frac{1}{2 \cdot 1} a_0 = -\frac{1}{2 \cdot 1} \qquad n=1$$

$$=4: a_4 = -\frac{1}{4 \cdot 5} a_2 = \frac{1}{(2 \cdot 4)(1 \cdot 5)} \qquad n=2$$

$$=6: a_6 = -\frac{1}{6 \cdot 9} a_4 = -\frac{1}{(2 \cdot 4 \cdot 6)(1 \cdot 5 \cdot 9)} \qquad n=3$$

$$a_{2n} = \frac{(-1)^n}{(2 \cdot 4 \cdot 6 \cdots 2n)(1 \cdot 5 \cdot 9 \cdots (4n-3))}$$

$${}^{-\frac{3}{2}}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n! (1 \cdot 5 \cdot 9 \cdots (4n-3))} x^{2n}\right) \qquad y = c_1 y_1 + c_2 y_2$$

$$= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 5(n+r)a_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r+1}$$ SHIFT DOWN BY 2

$$= \text{''} + \text{''} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

$$= 2r(r-1)a_0 x^{r-1} + 2(r+1)(r)a_1 x^r$$

$$+ 5r a_0 x^{r-1} \qquad + 5(r+1)a_1 x^r$$

$$+ \sum_{n=2}^{\infty} \left[2(n+r)(n+r-1)a_n + 5(n+r)a_n + a_{n-2}\right] x^{n+r-1}$$

$$= (2r(r-1)+5r)a_0 x^{r-1} + (2(r+1)r + 5(r+1))a_1 x^r$$

$$+ \sum_{n=2}^{\infty} \left[(n+r)(2n+2r\cancel{-2+5}^{+3})a_n + a_{n-2}\right] x^{n+r-1} = 0$$

$$(2r(r-1)+5r)a_0 = 0$$

$$(2(r+1)r+5(r+1))a_1 = 0$$

$$a_n = -\frac{1}{(n+r)(2n+2r+3)} a_{n-2}$$

FOR $n \geq 2$

FIND r UPFRONT:

$$y'' + p(x)y' + q(x)y = 0$$

EITHER p, q OR BOTH ARE DISCONT. @ $x = x_0$

SINGULAR POINT

BUT $(x-x_0)p(x)$, $(x-x_0)^2 q(x)$ ARE ANALYTIC @ $x = x_0$

REGULAR SINGULAR POINT

INDICIAL EQUATION

$$r(r-1) + ar + b = 0$$

WHERE $a = \lim_{x \to x_0} (x-x_0)p(x)$

$b = \lim_{x \to x_0} (x-x_0)^2 q(x)$

NOTE: IN OUR PROBLEM

$$\lim_{x \to 0} x p(x) = \lim_{x \to 0} x \cdot \frac{5}{2x} = \frac{5}{2}$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \cdot \frac{1}{2} = 0$$

$$r(r-1) + \frac{5}{2}r + 0 = 0$$

$$2r(r-1) + 5r = 0$$

PPLICATIONS

POPULATION

POPULATION GROWS AT RATE PROPORTIONAL TO EXISTING POPULATION

BAD ASSUMPTION

$$\frac{dP}{dt} = kP, \quad P(0)=P_0 \longrightarrow P = P_0 e^{kt}$$

$k>0$

AS $t \to \infty$, $P \to \infty$

EXPECT WHEN P IS LARGE
k IS SMALLER $\longrightarrow$ k IS NOT A CONSTANT
k IS A DECREASING FUNCTION OF P

$$\frac{dP}{dt} = k(P)\,P$$

$$\int \frac{1}{k(P)P}\,dP = \int dt$$

SIMPLEST

$k(P) = -mP + b$

$(m>0, b>0)$